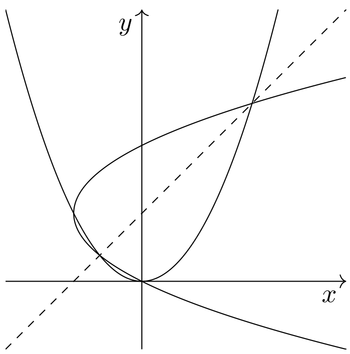
3801. A region R_k of the (x, y) plane is defined as the set of points that simultaneously satisfy $x^2 + y^2 \leq 1$ and $y \geq k|x|$, for some positive constant k. Show that, working in radians, the perimeter of R_k is

$$P_k = 2 + \pi - 2 \arctan k.$$

3802. In the diagram below, the curve $y = x^2$ has been reflected in the line $y = x + k$.



Find, with coefficients in terms of the constant k, the equation of the image.

3803. State, with a reason, which if any of the symbols $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a polynomial function f:

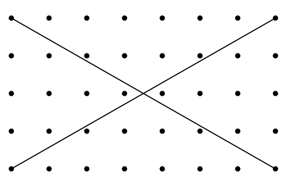
① $\mathrm{f}(x)$ has a factor of $(x - \beta)^3$,

② $y = \mathrm{f}(x)$ has a point of inflection at $x = \beta$.

3804. A resultant force $F = 5t - \frac{1}{2}t^2$ acts on an object of mass 500 grams, for $t \geq 0$. Determine the exact set of values of t for which the magnitude of the acceleration is greater than 21 ms^{-2}.

3805. The curve $x = y^4 - y^2$ is transformed by a stretch with the x axis invariant and a stretch with the y axis invariant. The image is $x = 4y^4 - 8y^2$. Find the area scale factor of the transformation.

3806. Prove by contradiction that, if an $m \times n$ lattice is such that its diagonals do not pass through any lattice points, then $\text{hcf}(m, n) = 1$.



3807. A curve is defined implicitly by $x^2 + y^3 = 1$.

(a) Find the equations of the tangents to the curve at $x = -1, 0, 1$.

(b) By considering behaviour as $x \to \pm\infty$, sketch the curve.

3808. Using a substitution, or otherwise, find

$$\int \frac{1 - 2e^{-x}}{1 + 2e^{-x}}\,dx.$$

3809. Find the largest real domain and codomain over which $\mathrm{f}(x) = x \ln x$ is an invertible function.

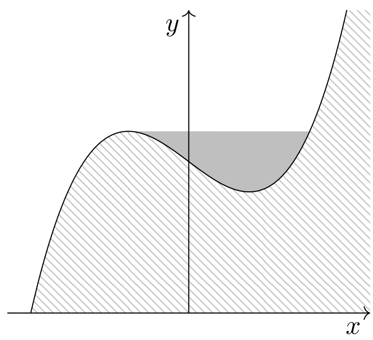
3810. In *tetrahedral molecular chemistry*, a central atom at O is surrounded by four others which lie at the vertices V_i of a tetrahedron. Show that such a structure generates a *bond angle* $\angle V_i O V_j$, for $i \neq j$, of approximately 109.5°.

3811. In this question, do not use a calculator.

Water is flowing down an ornamental cascade. The cascade is modelled with the equation

$$h = 10 + x^3 - 3x.$$

Variables h and x, in metres, are vertical height and horizontal position. In the y direction (also horizontal, into the page), the cross-section of the cascade is constant.



Show that the deepest pool that can form on the cascade has cross-sectional area 6.75 m^2.

3812. Simplify $\displaystyle\lim_{h\to 0} \frac{(2x + h)^n - (2x)^n}{(x + h)^n - x^n}$, for $x \neq 0$, $n \in \mathbb{N}$.

3813. The equations below define three parabolae. Show that they are pairwise tangent, and hence sketch them on the same set of axes.

$$y = x^2 + x,$$
$$x = y^2 + y,$$
$$x + y = (x - y)^2 + 2.$$

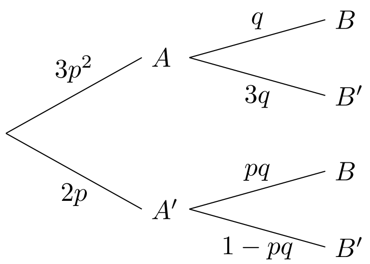
3814. The function f is quadratic, and $\mathrm{f}(x) > 0$ for all x. The quantity Q is defined as follows, as a function of the variable p:

$$Q = \int_p^{p+1} \mathrm{f}(x)\,dx.$$

You are given that Q is minimised at $p = 5/2$. Find the x coordinate of the vertex of $y = \mathrm{f}(x)$.

3815. Determine the number of different ways of placing n counters on an n-by-n grid, with a maximum of one per grid square, such that they are collinear.

3816. Events A and B have probabilities as represented on the following tree diagram, where $p, q \in \mathbb{R}$:



Draw a tree diagram conditioned on B, calculating all of the branch probabilities.

3817. Some data points (x, y) have been gathered in a statistical procedure. It is suspected that there is a quadratic relationship of the form $y = ax^2 + bx$ between the variables.

(a) Explain why a statistician might choose to plot the variable y/x against x.

(b) The line of best fit, when y/x is plotted against x, goes through the $(x, y/x)$ points $(0, 2.42)$ and $(4, 1.87)$. In the form $y = ax^2 + bx$, find the corresponding quadratic relationship.

3818. Solve for the variable z, given that both of the following equations are satisfied:

$$x^2 + y^2 - z^2 = 0,$$
$$x^2 + z = 12 - y^2.$$

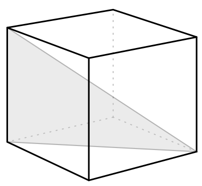
3819. A variable has distribution $X \sim B(8, p)$. It is given that $\mathbb{P}(X = 8 \mid X \geq 7) = \frac{1}{25}$. Find p.

3820. A function f has the following form, for non-zero coefficients $a_1, ..., a_k \in \mathbb{R}$ and $k \in \mathbb{N}$:

$$\mathrm{f}(x) = a_1x^4 + a_2x^6 + ... + a_{k-1}x^{2k}.$$

Prove that the second derivative is zero at $x = 0$, but that $y = \mathrm{f}(x)$ is not inflected there.

3821. The diagram shows a cube.



Three distinct vertices of the cube are chosen at random. Find the probability that the triangle so formed is congruent to the triangle depicted.

3822. A student is using the trapezium rule to estimate

$$\int_1^2 x \ln x \, dx.$$

Prove that this will overestimate the value of the integral, irrespective of the number of strips used.

3823. The line through $(p-1, 2p-1)$ and $(p^2-1, 4p^2-1)$ crosses the x axis at $x = 3$. Find p.

3824. A *perfect number* is a positive integer which is equal to the sum of all its positive divisors except itself. Let P be a perfect number, with its divisors (excluding P) as $d_1 < d_2 < ... < d_n$.

(a) Simplify $\dfrac{P}{d_1}$.

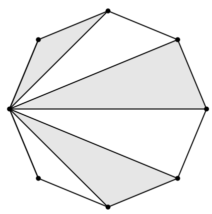
(b) Simplify $\dfrac{P}{d_k}$, for $k \in \{2, 3, ..., n\}$.

(c) Hence, prove that the sum of the reciprocals of the positive divisors (including P) of a perfect number is 2.

3825. Three dice have been rolled, giving scores X, Y, Z. Show that $\mathbb{P}(XY = Z \mid X + Y + Z = 5) = 1/3$.

3826. Sketch $y = \log_2(x^2 + 1)$, given that the graph is convex for $|x| < 1$ and convex for $|x| > 1$.

3827. In the diagram below, three shaded triangles have been constructed inside a regular octagon.



Prove that the areas of the two smaller triangles add to the area of the larger triangle.

3828. In this question, do not use a polynomial solver.

Using a numerical method, show that the quintic curve $y = x^5 - 3x^2 + 3x$ has two stationary points that lie close together.

3829. True or false?

(a) $\cos x \equiv |\cos x|$,

(b) $\cos(x^2) \equiv |\cos(x^2)|$,

(c) $\cos(x^3) \equiv |\cos(x^3)|$.

3830. A family of (x, y) curves C_a is defined by

$$y = x - 2a\sqrt{x} + a^2, \quad a \neq 0.$$

Show that

(a) some curves C_a are tangent to the x axis,

(b) every curve C_a is tangent to the y axis.

3831. A particle moves along a helix, with position at time t defined by

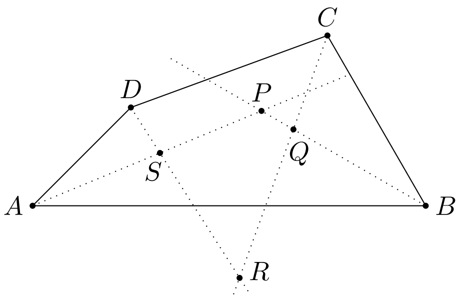
$$\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}.$$

(a) Show that the trajectory lies on the surface of the cylinder $x^2 + y^2 = 1$.

(b) Determine the constant speed of the particle.

(c) Show that the acceleration is always towards the axis of symmetry of the cylinder.

3832. In a quadrilateral $ABCD$, the angle bisectors are drawn. The points of intersection of bisectors at adjacent vertices are labelled P, Q, R, S.



Prove that $PQRS$ is a cyclic quadrilateral.

3833. Either prove or disprove the following statement: “For every $n \in \mathbb{N}$, there are systems of n linear equations in variables $x_1, ..., x_n$ which are satisfied by infinitely many $(x_1, ..., x_n)$ solution points.”

3834. Prove the following identity:

$$\frac{1}{\cos^2 x - \cos^4 x} \equiv 4 \operatorname{cosec}^2 2x.$$

3835. Sketch the following graphs on a single set of axes, marking any intercepts.

① $y = x^5 - x$,
② $y = x^7 - x$.

3836. In this question, do not use a polynomial solver.

A quartic function is defined as

$$\mathrm{f}(x) = 27x^4 + 54x^3 - 180x^2 + 136x - 32.$$

You are given that f has exactly two roots, and that f is stationary at exactly one of these roots.

(a) By considering the multiplicity of the roots, explain how you know that the stationary root is a point of inflection.

(b) Solve $\mathrm{f}''(x) = 0$.

(c) Hence, determine the solution of $\mathrm{f}(x) = 0$.

3837. Show that ABRACADABRA has 83160 anagrams.

3838. A projectile, launched from ground level, has a flight in which the maximum height attained is exactly half the range.

Show that the projectile was launched at an angle $\theta = \arctan 2$ above the horizontal.

3839. Determine the value of $\displaystyle\lim_{x\to 0} \frac{16^x - 1}{4^x - 1}$.

3840. Consider the parabola $x = y^2 - 2y + 3$.

(a) Find the coordinates of the vertex.

(b) Sketch the parabola.

(c) Find the area enclosed by the parabola and the line $x = 3$.

3841. Prove that the number of points of tangency of two circles can only be zero, one, or infinite.

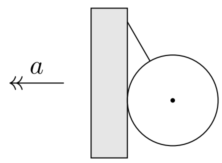
3842. An iteration is given by

$$x_{n+1} = 1 - \frac{x_n}{2 + x_n}.$$

Show that this iteration cannot be periodic with period 2, whatever the starting value.

3843. Show that the area of the region enclosed by the graphs $y = |x| + 3$ and $y = 20 - 4|x - 4|$ is 45.

3844. A smooth uniform sphere of radius r and mass m is hung from a point on a movable block by a light, inextensible string of length r. The movable block then begins to accelerate to the left. The sphere does not rotate.



(a) Explain why the line of action of the tension must pass through the centre of the sphere.

(b) Determine the greatest acceleration a_{max} for which the sphere will remain in contact with the block.

3845. Numbers x and y are chosen at random from the interval $[0, 1]$. Show that, for $k \in [0, 1]$,

$$\mathbb{P}\big(|x - y| < k\big) = 2k - k^2.$$

3846. The second parametric derivative is given by

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

(a) Show that the curve $x = \cos t$, $y = \cos 2t$ has constant second derivative.

(b) Hence, state what type of curve this is.

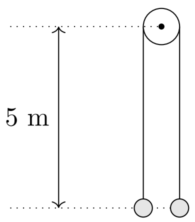
3847. Sketch $y = \left(4x^2 - 5x + 1\right)^6$.

3848. Prove that, for any sample $\{x_i\}$ of size n,

$$\sum_i (x_i - \bar{x})^3 \equiv \sum_i x_i^3 - 3\bar{x}\sum_i x_i^2 + 2n\bar{x}^3.$$

3849. In this question, take $g = 10$.

Two bobs, masses m and $3m$ kg, are attached by a light 10 metre rope which is passed over a small smooth pulley. The bobs are held in equilibrium, hanging at the same vertical height of 5 metres below the pulley.



The bobs are then released.

(a) Find the positions and velocities of the bobs one second after release.

(b) One second after release, the pulley snaps, and the bobs become projectiles. Find the total time elapsed when the rope next becomes taut.

3850. A curve is defined implicitly by

$$(2x-1)(2y-1) + 4x^2y^2 = 0.$$

Show that the line $x+y=0$ is tangent to the curve at two distinct points.

3851. Trigonometric calculus is usually done in radians. In this question, however, the unit of angle is the degree. Determine the equation of the tangent to $y = \sin x$ at the origin.

3852. A *Lissajous curve* is a graph given parametrically as follows, for some $k_1, k_2 \in \mathbb{N}$:

$$x = \sin k_1 t,$$
$$y = \sin k_2 t.$$

Determine the t-period, with t defined in radians, of the Lissajous curves with

(a) $k_1 = 2$, $k_2 = 3$,

(b) $k_1 = 8$, $k_2 = 12$.

3853. Using a substitution, or otherwise, show that

$$\int_{\pi^2}^{4\pi^2} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = 0.$$

3854. Prove that $\log_a xy \equiv \log_a x + \log_a y$.

3855. Let f be a function with the following properties:

① $\mathrm{f}(x) = 0$ has a root at $x = \alpha$,

② f is decreasing everywhere on its domain.

Provide a counterexample to the following claim:

"If the Newton-Raphson iteration is used to solve $\mathrm{f}(x) = 0$, then it will necessarily find the root $x = \alpha$, whatever starting value is used."

3856. Curve C has equation

$$y = x^2 + \frac{x-1}{x+1}.$$

A straight line L with gradient m crosses C at $(1,1)$, and is a tangent to it elsewhere.

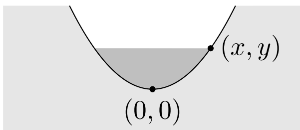
(a) Show that the x coordinates of intersections of C and L satisfy

$$x^3 + (1-m)x^2 - 2 + m = 0.$$

(b) Hence, find the two possible equations of L.

3857. 20% of the claims made to a particular insurance company are fraudulent. Of the fraudulent claims, only 45% are rejected as such, while 25% of the genuine claims are rejected as being fraudulent. Find the probability that a rejected claim is, in fact, genuine.

3858. An irrigation ditch is modelled as having constant cross-section in the shape of the parabola $y = x^2$. Water evaporates from the ditch at a rate which is proportional to the surface area of the water.



For a period of time, there is no flow in or out of the irrigation ditch.

(a) Show carefully that $V \propto x^3$.

(b) Hence, show that the depth of water in the ditch decreases at a constant rate.

3859. By factorising, solve, for $\theta \in [-\pi, \pi]$,

$$\sin 4\theta - 2\sin 2\theta + \cos 2\theta - 1 = 0.$$

3860. Prove that there is no equilateral triangle which has integer values for both perimeter and area.

3861. A curve has equation $x^3 + \sqrt{y} = 1$.

(a) Find and classify any stationary points.

(b) Hence, sketch the curve.

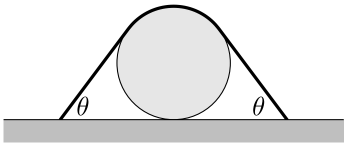
3862. Two *coupled differential equations* are given by

$$\frac{dx}{dt} = x - y, \quad \frac{dy}{dt} = x + y.$$

Verify that, for constant $A \in \mathbb{R}$ and variable $t \in \mathbb{R}$, the parametric curve $x = Ae^t \cos t$, $y = Ae^t \sin t$ satisfies both equations.

3863. The parabola $y = x^2 + ax + b$ has a minimum at (p, q). Find, in the form $y = -x^2 + cx + d$, the equation of a parabola which has a maximum at $(-p, -q)$.

3864. A smooth cylindrical barrel of radius 50 cm and weight W is fastened down to the deck of a ship with a rope. The tension in the rope is $\frac{5}{2}W$, and the points of attachment to the deck are 2 m apart.



(a) Show that $\sin\theta = \frac{4}{5}$.

(b) Draw a force diagram for the barrel.

(c) Determine, in terms of W, the magnitude of the contact force between deck and barrel.

3865. A function is defined over $\mathbb{R}$ by

$$f(x) = x - 2x^{\frac{3}{5}} + x^{\frac{1}{5}}.$$

(a) Show that $f'(0)$ is undefined.

(b) Determine the two x values for which

$$f'(x) = f(x) = 0.$$

3866. A particle moves with position at time t

$$x = (1 - \cos t)\cos t,$$
$$y = (1 - \cos t)\sin t.$$

Show that the path of the particle has a single *cusp*, i.e. that there is a single (x, y) location at which the particle reverses its direction of travel.

3867. Find the exact area of the elliptical region of the (x, y) plane defined by $ax^2 + by^2 \le ab$.

3868. Without doing any calculations, sketch the graph $y = (x - a)^n(x - b)^n(x - c)^n$, where $0 < a < b < c$, in the cases that

(a) $n = 1$,

(b) $n = 2$,

(c) $n = 3$.

3869. Prove that $\dfrac{d}{dx}(\arccos x) = -\dfrac{1}{\sqrt{1 - x^2}}$.

3870. By expanding $(1 - x)^{\frac{1}{3}}$, and using the fact that $0.027 \times 37 = 0.999$, show that $\sqrt[3]{37} \approx \frac{2999}{900}$.

3871. The following differential equation is known as an *Euler-Cauchy* equation:

$$t\frac{d^2x}{dt^2} + \frac{dx}{dt} = 0.$$

(a) Verify that $x = A\ln t + B$ satisfies the DE, for any constants $A, B \in \mathbb{R}$.

(b) The position of a moving particle is modelled by the solution in part (a). According to the model, find the percentage change in the speed of the particle between $t = 1$ to $t = 5$.

3872. A parametric curve has equations

$$x = \sin 3t,$$
$$y = \sin t.$$

Using one or more trigonometric identities, find the Cartesian equation of the curve, giving x in simplified polynomial terms of y.

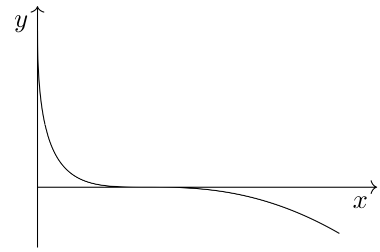
3873. Two events A and B are such that $\mathbb{P}(A \mid B) = \frac{1}{5}$ and $\mathbb{P}(B \mid A) = \frac{4}{5}$. Let $\mathbb{P}(A \cap B) = k$.

(a) Show that $\mathbb{P}(B) = 4\,\mathbb{P}(A)$.

(b) Draw a Venn diagram, writing the probability of each outcomes in terms of k.

(c) Hence, find all possible values of $\mathbb{P}(A \cap B)$.

3874. The curve $y = \left(1 - \sqrt{x}\right)^3$ is as shown:



A straight line is drawn, which passes through the axis intercepts. Show that this line intersects the curve again at $(9, -8)$.

3875. In this question, do not use a polynomial solver.

A quartic function h is defined as

$$h(x) = 4x^4 - 4x^3 + 5x^2 - 4x + 1.$$

You are given that the quartic equation $h(x) = 0$ has exactly one root, $x = \alpha$.

(a) Explain why α must be a repeated root.

(b) Find this root.

(c) Hence, factorise $h(x)$ fully.

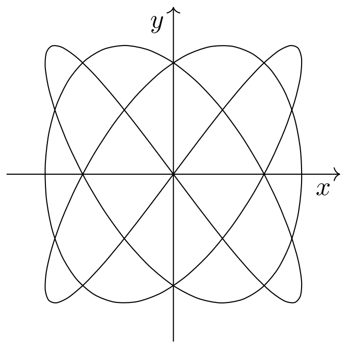
3876. Prove that, for small x, $\cos 2x \approx \sqrt{1-4x^2}$.

3877. Sketch the region of the (x, y) plane which satisfies

$$\big(x - |x|\big)\big(y - |y|\big) > 0.$$

3878. The diagram shows a parametric *Lissajous* curve, defined for $t \in [0, 2\pi)$. Its equations are

$$x = \sin 3t,$$
$$y = \sin 4t.$$



Find the exact coordinates of the six points at which the tangent to the curve is parallel to the y axis.

3879. Prove that, if f is a cubic function, then the curves $y = \mathrm{f}(x)$ and $x = \mathrm{f}(y)$ have at least one point of intersection.

3880. A sequence is given by $u_{n+2} = u_{n+1} - u_n$. Show that, whatever the starting values u_1 and u_2, the sequence is periodic.

3881. Show that $\displaystyle\int_0^3 \frac{3x+1}{(x+1)(x+3)}\,dx = \ln 4$.

3882. A graph is defined by $\cos ay + \cos bx = 0$, where a and b are positive constants. This graph partitions the (x, y) plane into infinitely many regions, which are all congruent.

(a) Show that, in the case $a = b = 1$, the region containing the origin has area $2\pi^2$.

(b) Hence, or otherwise, show that, in the general case, the regions have area

$$A = \frac{2\pi^2}{ab}.$$

3883. A function is defined as $\mathrm{f}(x) = e^x + e^{-x}$. Show that, for some constant k to be determined,

$$\mathrm{f}(2x) \equiv \big(\mathrm{f}(x)\big)^2 + k.$$

3884. By factorising, sketch the set of (x, y) points which satisfy the equation

$$x^3 - x^2y + 2xy - 2y^2 = 0.$$

3885. Two maths students are discussing the application of integration by parts to

$$\int x^2 \ln x \, dx.$$

One says "We should use $u = x^2$." The other says "No. We should use $u = \ln x$." Explain how the method breaks down with the wrong choice for u, and perform the integration with the right choice for u.

3886. A tangent is drawn to the curve $y = x^2$ at a point $\big(p, p^2\big)$. Show that the area enclosed by the curve, the tangent and the x axis is given by $\frac{1}{12}p^3$.

3887. Solve $(\log_2 x)^2 - \log_4 x^6 - 4 = 0$.

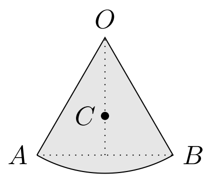
3888. Prove that the graph $y = x^4 + x^3$ does not have reflective or rotational symmetry.

3889. Two curves are defined by the equations

$$y = \sqrt{3}\sec^2 x,$$
$$y = 4\tan x.$$

These curves enclose infinitely many regions. Show that each regions has area $A = \ln 9 - 2$.

3890. A toy of mass m has the shape of a sector of a sphere. In cross-section, this gives a sector OAB of a circle, with 60° subtended at the centre. The centre of mass of the toy lies $\frac{2}{3}$ of the way from O to the midpoint of AB. The toy stands on horizontal ground.



A force P is now applied horizontally at O, such that the toy is tilted, in equilibrium, at an angle $\theta \in (0, 30°)$ away from the resting position shown above.

(a) Draw a force diagram for the toy.

(b) Show that $P = \frac{\sqrt{3}}{3}mg\sin\theta$.

3891. A cubic function f has a derivative satisfying

$$\mathrm{f}'(-2) = \mathrm{f}'(2) = 0.$$

Prove that the point of inflection of the cubic graph $y = \mathrm{f}(x)$ lies on the y axis.

3892. Events A and B are such that

$$\mathbb{P}(A \cup B) = x,$$
$$\mathbb{P}(A \cap B) = y.$$

Find the following in terms of x and y:

(a) $\mathbb{P}(A' \cup B')$,
(b) $\mathbb{P}(A \cap B \mid A \cup B)$,
(c) $\mathbb{P}\big((A \cup B') \cup (A' \cup B)\big)$.

3893. The graphs $y = x^{\frac{1}{n}}$ and $y = x^{\frac{1}{n-1}}$, where n is an integer greater than 1, are defined for $x \geq 0$. Show that the area A enclosed by the curves is given by

$$A = \frac{1}{n^2 + n}.$$

3894. (a) Prove the identity

$$\frac{1 - \sin 2x}{1 + \sin 2x} \equiv \frac{\sec^2 x - 2\tan x}{\sec^2 x + 2\tan x}.$$

(b) Hence, prove the identity

$$\sqrt{\frac{1 - \sin 2x}{1 + \sin 2x}} \equiv \left|\frac{1 - \tan x}{1 + \tan x}\right|.$$

3895. An AP has first term 4. Two partial sums are

$$S_n = 310,$$
$$S_{n+1} = 374.$$

Find the value of the first partial sum which is greater than 1000.

3896. By factorising, or otherwise, shade the regions of the (x, y) plane that satisfy $x + y - xy < 1$.

3897. A quintic function q is defined as

$$\mathrm{q}(x) = 1 + x + x^2 + x^3 + x^4 + x^5.$$

Factorise $\mathrm{q}(x)$ into linear and quadratic factors.

3898. Find $\displaystyle\int \frac{2x^5 + 5x^2}{x^3 + 1}\,dx$.

3899. An arithmetic progression is given as $\{p, q, r, s\}$, where all terms are positive. Show that such a set of lengths can be used to construct a quadrilateral.

3900. Find the probability that four consecutive rolls of a die give scores s_i such that $s_1 + s_2 = s_3 + s_4$.

——— End of 39th Hundred ———